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DENSITY AND UNIQUE DECOMPOSITION THEOREMS FOR THE LATTICE OF CELLULAR CLASSES

BY

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ABSTRACT

A class $\mathcal{C}$ of pointed spaces is called a cellular class if it is closed under weak equivalences, arbitrary wedges and pointed homotopy pushouts. The smallest cellular class containing X is denoted by $C(X)$, and a partial order relation $\ll$ is defined by: $X \ll Y$ if $Y \in C(X)$. In this text we investigate the sub partial order sets generated respectively by simply connected finite CW-complexes and by rational spaces. For rational spaces we prove a unique decomposition theorem, a density theorem and the existence of infinitely many non-comparable elements. We then prove the density theorem for a generic class of finite CW-complexes.

1. Introduction

Cellular classes were introduced by E. Dror Farjoun [8] in the context of Bousfield localization and colocalization theories [2]. The aim of this paper is the study of the properties of the lattice formed by cellular classes.

For the sake of simplicity we restrict ourselves to the family of strictly simply-connected CW-complexes, i.e., simply-connected CW-complexes with nontrivial π_2. Similar results can easily be obtained for the category of $(2n-1)$-connected finite CW-complexes. A class $\mathcal{C}$ of pointed spaces is called a **cellular class** if it satisfies the following three closure properties:

- (closure under weak equivalences) if $A \in \mathcal{C}$ and $B \simeq A$, then $B \in \mathcal{C}$;
- (closure under arbitrary wedges) if $\{A_k\} \in \mathcal{C}$, then $\bigvee_k A_k \in \mathcal{C}$; and

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- (closure under pointed homotopy pushouts) if $A, B, C \in \mathcal{C}$, then any pointed homotopy pushout $hocolim_*(B \leftarrow A \rightarrow C)$ belongs to $\mathcal{C}$.

The smallest cellular class containing a space X is denoted by $C(X)$, e.g., $C(S^2)$ consists of the simply-connected CW-complexes. A space X is said to **build** a space Y, if Y belongs to $C(X)$. In this case we write $X \ll Y$. If $X \ll Y$ and $Y \ll X$, then the two spaces are said to be **cellularly equivalent**, a relation that we denote by $X \sim Y$. A strict inequality $X \ll Y$ is denoted by $X \overset{s}{\ll} Y$. The **rational cellular class** of X is the cellular class $C(X_0)$ generated by the rationalization X_0 of X.

In [5], W. Chachólski, P. E. Parent and D. Stanley show that the partially ordered set (poset) $(Spaces, \ll)$ is a complete lattice. The meet of two spaces is the wedge of the spaces. The join of two simply-connected finite CW-complexes X and Y is the wedge of all simply-connected countable CW-complexes that are built by X and Y.

We denote by $\mathcal{B}$ the family of cellular classes of strictly simply-connected finite CW-complexes, and by $\mathcal{B}_0$ the corresponding family for strictly simply-connected c-finite rational spaces. Here a rational space is called c-finite (cohomologically finite) if $\sum_{i\geq 2} \dim H^i(X;\mathbb{Q}) < \infty$. In [6], Chachólski, Parent and Stanley prove the following Theorem that will be absolutely crucial throughout the paper.

THEOREM ([6]): *Let X and Y be strictly simply-connected rational spaces. Then $X \ll Y$ if and only if there is a continuous map $f\colon \bigvee_{i\in I} X \to Y$ such that $H_2(f)$ is surjective.*

The posets $\mathcal{B}$ and $\mathcal{B}_0$ contain infinitely many elements. For instance,

$$S^2 \overset{s}{\ll} P^2(\mathbb{C}) \overset{s}{\ll} P^3(\mathbb{C}) \overset{s}{\ll} \cdots \overset{s}{\ll} P^\infty(\mathbb{C}); \quad \text{and}$$
$$S^2 \overset{s}{\ll} \cdots \overset{s}{\ll} P^2(\mathbb{C})\#P^2(\mathbb{C})\#P^2(\mathbb{C}) \overset{s}{\ll} P^2(\mathbb{C})\#P^2(\mathbb{C}) \overset{s}{\ll} P^2(\mathbb{C}).$$

Clearly, the structure of these posets is highly nontrivial as shown by the following results.

In [14] K. Hess has shown that the poset $\mathcal{B}_0$ contains non-comparable elements that have the same rational cohomology algebra and the same rational homotopy Lie algebra.

More recently an injection of the usual ordered set $(\mathbb{R}, \leq)$ into the poset of (non-necessarily finite dimensional) rational spaces has been obtained by Chachólski, Parent and Stanley ([7]).

In this paper we consider the following important problem, namely, *Density problem:* Let X and Y be simply-connected finite CW-complexes such that

$$S^2 \ll X \overset{s}{\ll} Y \ll K(\mathbb{Q}, 2).$$

Does there exist a simply-connected finite CW-complex Z not equivalent to X and Y such that $X \ll Z \ll Y$?

Our first theorem gives an affirmative answer to the density problem for the poset $\mathcal{B}_0$. We also show that there are infinitely many non-comparable elements in this poset.

THEOREM 1 (Rational density): *Suppose that $X \overset{s}{\ll} Y$ is a strict inequality in $\mathcal{B}_0$. Then there are infinitely many non-comparable strictly simply-connected c-finite rational spaces Z_n, $n \geq 1$, such that $X \overset{s}{\ll} Z_n \overset{s}{\ll} Y$.*

Finally, we construct injections of the usual poset $(\mathbb{Q} \cap [0,1], \leq)$ into $\mathcal{B}$.

THEOREM 2: *There is a family of simply connected finite CW-complexes $\mathcal{F}$ that satisfies the following property. If $X \in \mathcal{F}$, and if ω is an element in $\pi_{2q}(\Omega X)$ with $2q+1 \geq \dim X$, which is not in the radical $R(X)$ of X, and such that $\omega \odot 1$ is nontrivial in $\pi_{2q}(\Omega X) \odot \mathbb{Q}$, then there exists a poset injection*

$$f_{X,\omega}\colon (\mathbb{Q} \cap [0,1], <) \to (\mathcal{B}, \ll),$$

with $f_{X,\omega}(0) = X$ and $f_{X,\omega}(1) = (X \cup_{\tilde{\omega}} e^{2q+2})$. Here $\tilde{\omega}$ denotes the element of $\pi_{2q+1}(X)$ corresponding to ω by the natural adjunction.

Since $(\mathbb{Q}, \leq)$ injects into $(\mathbb{Q} \cap]0,1[, \leq)$, Theorem 2 gives different injections of $(\mathbb{Q}, \leq)$ into the poset $(\mathcal{B}, \ll)$. In order to describe the family $\mathcal{F}$, we need to introduce some definitions.

Definition 1: A space X is called **(rationally) irreducible** if its rationalization X_0 satisfies the following properties.

- (I_1) X_0 is not equivalent to a wedge of rational spaces $\bigvee_{i \in I} X_i$ with $\dim H_2(X_i) < \dim H_2(X_0)$ for $i \in I$; and
- (I_2) a self-map f of X_0 that induces an isomorphism on $H_2(X_0)$ is a homotopy self-equivalence.

For example, a space whose rational cohomology algebra is generated by elements of degree 2 satisfies I_2. More generally, spaces X for which the rational Hurewicz map $h_q\colon \pi_q(X) \odot \mathbb{Q} \to H_q(X; \mathbb{Q})$ is zero for $q > 2$ also satisfy I_2 (cf. Proposition 3, below). When n is even, a connected sum of r copies of $P^n(\mathbb{C})$,

$X = \#^r P^n(\mathbb{C})$, is irreducible. Suppose in fact that X_0 is cellularly equivalent to a wedge $Y = \bigvee_{i\in I} X_i$. This implies the existence of maps g and h

$$\bigvee_k X_0 \xrightarrow{g} \vee_j Y \xrightarrow{h=\vee_j h_j} X_0$$

such that $H^2(h)$ and $H^2(g)$ are injective. Suppose $\dim H_2(X_i) < \dim H_2(X_0)$; then for $i \in I$ and $j \in J$ there is some $\alpha_{ij} \in H^2(X_0)$ such that $H^2(h_j|_{X_i})(\alpha_{ij}) = 0$. Since n is even, $H^{2n}(X_0)$ is generated by α_{ij}^n. This implies that $H^{2n}(h) = 0$. Take $\beta \in H^2(X_0)$. We have $H^{2n}(h)(\beta^n) = 0$. But this is impossible because, from the injectivity of $H^2(h\circ g)$, we have $H^{2n}(h\circ g)(\beta^n) = (H^2(h\circ g)(\beta))^n \neq 0$.

Similar arguments show that $P^3(\mathbb{C})\#(S^3 \times S^3)$ is another example of an irreducible space.

Definition 2 ([10]): A simply-connected finite CW-complex X is called **(rationally) hyperbolic** (respectively **elliptic**) if the graded vector space $\pi_*(X)\otimes\mathbb{Q}$ is infinite dimensional (resp. finite dimensional).

The dichotomy between elliptic and hyperbolic spaces is very important in rational homotopy theory. If X is elliptic, then its rational cohomology algebra $H^*(X;\mathbb{Q})$ satisfies Poincar duality, the Euler–Poincaré characteristic is non-negative, and $\pi_q(X)\otimes\mathbb{Q} = 0$ for $q \geq 2\cdot\dim X$. On the other hand, when X is hyperbolic, the graded Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ is not nilpotent, and the union $R(X)$ of all solvable ideals in $\pi_*(\Omega X)\otimes\mathbb{Q}$, called the **radical** of X, is a finite dimensional nilpotent Lie algebra ([9], ([10], Theorem 36.5)). Moreover, $\dim R_{even}(X) \leq \text{cat}\,(X)$ ([10], Theorem 36.5), where cat (X) denotes the Lusternik–Schnirelmann category of the space X ([18]), where by convention the category of a contractible space is zero. In particular, if $\omega \notin R(X)$, then the ideal generated by ω is infinite dimensional.

Definition 3: A **Bousfield class** $\mathcal{C}$ is a cellular class together with the requirement that whenever $F \to E \to B$ is a fibration sequence in which $F, B \in \mathcal{C}$, then $E \in \mathcal{C}$. In [5], the authors show that any Bousfield class generated by a space has a cellular generator. When two spaces generate the same Bousfield class, they are called Bousfield equivalent. For instance, a deep result of Hopkins and Smith ([16]) shows that if X is a simply-connected finite CW-complex, and $\pi_2(X) = \mathbb{Z}\oplus G$, then X is Bousfield equivalent to S^2.

We can now make Theorem 2 more precise. The family $\mathcal{F}$ is the family of irreducible, hyperbolic, finite, simply-connected CW-complexes that are Bousfield equivalent to the sphere S^2.

Let us come back to the rational poset. In order to prove the rational density we first prove a unique decomposition theorem.

Definition 4: A decomposition $X \sim (\bigvee_{i\in I} X_i) \vee Y$ is called **an irreducible decomposition relative to** Y if the following properties are satisfied:

(P_1) each X_i is an irreducible space; and

(P_2) if for some $i_0 \in I$ we have a continuous map

$$f = \bigvee_k f_k\colon \bigvee_k \left(\left(\bigvee_{i\in I} X_i\right) \vee Y\right) \to X_{i_0}$$

such that $H_2(f;\mathbb{Q})$ is surjective, then for some k, the restriction $f_{k|X_{i_0}}: X_{i_0} \to X_{i_0}$ is a rational homotopy equivalence.

When Y is contractible, the irreducible decomposition relative to Y is called **an irreducible decomposition** of X. An irreducible space X is an irreducible decomposition of X (cf. Proposition 1 below).

THEOREM 3: *Let X and Y be strictly simply-connected c-finite rational spaces; then $X \vee Y$ admits an irreducible decomposition relative to Y.*

THEOREM 4 (Unique decomposition theorem): *A strictly simply-connected c-finite rational space admits a unique (up to permutation of the factors) irreducible decomposition.*

In other words, if $\bigvee_{i\in I} X_i$ and $\bigvee_{j\in J} Y_j$ are irreducible decompositions, then there is a bijection $f\colon I \to J$ such that $X_i \sim Y_{f(i)}$.

On one hand, the poset $\mathcal{B}_0$ is rather simple (unique decomposition theorem), on the other, it seems very complicated. In particular, we construct (Theorem 5) an injection θ of the poset of finitely generated ideals in a free graded Lie algebra on two generators into $\mathcal{B}_0$ satisfying $\theta(I) \ll \theta(J)$ if and only if $I \subset J$.

The paper is organized as follows. In section 3 we develop properties of irreducible spaces and irreducible decompositions. Section 4 contains tools for the construction of strict inequalities. Sections 5 and 6 are devoted to the proof of Theorem 1, and Section 7 to the proof of Theorem 2.

The proof of Theorem 1 is based on the following argument. Suppose $X \overset{s}{\ll} Y$; then $X \sim X \vee Y \overset{s}{\ll} Y$. We then take an irreducible decomposition of $X \vee Y$ relative to Y. This gives

$$X \sim \bigvee_{i=1}^{n} X_i \vee Y \ll \bigvee_{i=2}^{n} X_i \vee Y \ll \cdots \ll X_n \vee Y \ll Y.$$

By assumption, one inequality has to be strict, thus there is a sequence

$$X \ll Z \vee T \overset{s}{\ll} T \ll Y,$$

with $Z \vee T$ irreducible relative to T.

Now the proof splits into two cases depending on whether Z is hyperbolic or elliptic. In the elliptic case, let n be the least integer m such that $\pi_{\geq 2m}(Z) \odot \mathbb{Q} = 0$. We choose an element $\alpha \in \pi_{2n-1}(Z)$ such that $\alpha \odot 1 \neq 0$ in $\pi_{2n-1}(Z) \odot \mathbb{Q}$. Let N be an integer greater than the cohomological dimension of $Z \vee T$, and let M be an integer greater than $2 \cdot \dim H_n(Z \vee T; \mathbb{Q})$. Depending on the value of n, we consider the following space R_Z:

$$\text{when } n = 2,\ R_Z = \left(\left(\bigvee_{i=1}^{2M} P_i^N(\mathbb{C}) \vee Z \right) \cup_{\alpha+\omega} e^{2n} \right)_0, \quad \omega = \sum_{i=1}^{M} [\beta_{2i-1}, \beta_{2i}]$$

and

$$\text{when } n > 2,\ R_Z = \left(\left(\bigvee_{i=1}^{2M} S_i^n \vee Z \right) \cup_{\alpha+\omega} e^{2n} \right)_0, \quad \omega = \sum_{i=1}^{M} [S_{2i-1}^n, S_{2i}^n],$$

where the $P_i^N(\mathbb{C})$ are copies of the complex projective space $P^N(\mathbb{C})$, and β_i is a generator of $\pi_2(P_i^N(\mathbb{C}))$.

We show that $R_Z \vee T$ is irreducible relative to T, that R_Z is hyperbolic, and that

$$Z \vee T \overset{s}{\ll} R_Z \vee T \overset{s}{\ll} T.$$

Finally, we prove that if $S \vee T$ is irreducible relative to T and S is hyperbolic, then there is a sequence of non-comparable spaces Z_m such that

$$S \vee T \overset{s}{\ll} Z_m \overset{s}{\ll} T.$$

Throughout this paper, if α is an element of $\pi_q(\Omega X)$, then $\tilde{\alpha}$ denotes the corresponding element in $\pi_{q+1}(X)$ through the natural isomorphism $\pi_q(\Omega X) \cong \pi_{q+1}(X)$.

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2. Backgrounds

2.1. Overview of rational homotopy theory. In this text we mainly use tools from rational homotopy theory, and refer the reader to [21], [10] and [22] for the necessary background. Each simply-connected space with finite Betti numbers admits a Sullivan minimal model $\mathcal{M}_X = (\wedge V, d)$ and a Quillen minimal model $\mathcal{L}_X = (\mathbb{L}(W), d)$ that have the following properties:

$$\mathrm{Hom}_{\mathbb{Q}}(V^n, \mathbb{Q}) \cong \pi_n(X) \otimes \mathbb{Q} \cong H_{n-1}(\mathbb{L}(W), d), \quad \text{and}$$
$$H^n(\wedge V, d) \cong H^n(X; \mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(W_{n-1}, \mathbb{Q}).$$

Each continuous map $f: X \to Y$ admits a Sullivan minimal model $\mathcal{M}_f: \mathcal{M}_Y \to \mathcal{M}_X$ and a Quillen minimal model $\mathcal{L}_f: \mathcal{L}_X \to \mathcal{L}_Y$. Moreover, each morphism of differential graded algebras $\varphi: \mathcal{M}_Y \to \mathcal{M}_X$ or differential graded Lie algebras $\psi: \mathcal{L}_X \to \mathcal{L}_Y$ can be realized by a continuous map $f: X_0 \to Y_0$. Denote by $[-,-]$ the set of homotopy classes. There are bijections

$$[X_0, Y_0] \cong [\mathcal{M}_Y, \mathcal{M}_X] \cong [\mathcal{L}_X, \mathcal{L}_Y].$$

The dichotomy between elliptic and hyperbolic spaces, as noted in the introduction, is very useful. For instance, in the elliptic case, the greatest integer m such that $\pi_m(X) \neq 0$ is always odd ([11]); and the rational homotopy Lie algebra of an hyperbolic space has exponential growth.

A simply-connected c-finite rational space Y is always the rationalization of a simply-connected finite CW-complex X, ($Y = X_0$). From the cell decomposition of X we deduce a rational cell decomposition of X_0. In fact, if $X = Z \cup_\alpha e^n$ then X_0 is the homotopy cofiber of the map $\alpha_0: S_0^{n-1} \to Z_0$, and we say that Y is obtained from Z_0 by adjunction of a rational cell of dimension n. The (rational) skeleton of dimension p of Y, denoted Y^p, is the subspace of Y formed by the rational cells of dimension $\leq p$. This procedure shows how to construct, in a very simple way, the rationalization of a space X from a cellular decomposition of X. By a result of Baues ([1]), we can always choose a cell decomposition in which the suspensions of the attaching maps of the cells are trivial.

2.2. Posets and lattices. A **lattice** $(L, \vee, \cap)$ is a nonempty set closed under two binary operations $\vee$ (join) and $\cap$ (meet) such that the following laws are satisfied for all $a, b, c \in L$:

- associative laws: $a \vee (b \vee c) = (a \vee b) \vee c, a \cap (b \cap c) = (a \cap b) \cap c$;
- commutative laws: $a \vee b = b \vee a, a \cap b = b \cap a$; and
- absorption laws: $a \vee (a \cap b) = a, a \cap (a \vee b) = a$.

An order relation $\leq$ can be defined on a lattice such that $a \leq b$ means that $a \vee b = b$.

A lattice can be seen as a poset $(E, \leq)$ such that any set of two elements possesses both a least upper bound and a greatest lower bound.

3. Irreducible spaces

3.1. IRREDUCIBILITY PROPERTIES AND CRITERIA. We call a map $f\colon X \to Y$ H_n-surjective or injective if the map $H_n(f)$ is respectively surjective or injective.

LEMMA 1: *Let $f = \bigvee f_i\colon \bigvee_{i\in I} X_i \to Y$ be an H_2-surjective map between simply-connected c-finite rational spaces. Then each f_i factors as $X_i \xrightarrow{h_i} Y_i \xrightarrow{g_i} Y$ such that the h_i are H_2-surjective, the g_i are H_2-injective, the map $\bigvee_i g_i\colon \bigvee_{i\in I} Y_i \to Y$ is H_2-surjective and the spaces Y_i are simply-connected c-finite rational spaces.*

Proof: Denote by $\varphi_i\colon (\wedge Z, d) \to (\wedge V_i, d)$ a Sullivan minimal model of f_i, and denote by (I_i, d) its image. Clearly (I_i, d) is a commutative differential graded algebra, and φ_i factors through it, i.e.,

$$\varphi_i = \theta_i \circ \psi_i\colon (\wedge Z, d) \xrightarrow{\psi_i} (I_i, d) \xrightarrow{\theta_i} (\wedge V_i, d).$$

We denote by Y_i the (Sullivan) geometric realization of the algebra (I_i, d) ([10], [21]). The algebra maps ψ_i and θ_i can be realized by maps $g_i\colon Y_i \to Y$ and $h_i\colon X_i \to Y_i$. Denote by n_i the cohomological dimension of X_i. Since $H^*(I_i, d)$ can be infinite dimensional, in order to satisfy all the requirements of the lemma, we replace Y_i by its rational skeleton of dimension n_i. ∎

PROPOSITION 1.: *An irreducible space X is an irreducible decomposition of X.*

Proof: Suppose $f = \bigvee_{i\in I} f_i\colon \bigvee_{i\in I} X \to X$ is an H_2-surjective map. By Lemma 1, each f_i factorizes as $X \xrightarrow{h_i} Y_i \xrightarrow{g_i} X$ such that the h_i are H_2-surjective, the g_i are H_2-injective, the map $\bigvee_i g_i\colon \bigvee_{i\in I} Y_i \to X$ is H_2-surjective and the spaces Y_i are simply-connected c-finite rational spaces. Then X is cellularly equivalent to $\bigvee_i Y_i$. By irreducibility property I_1, for some i, $H_2(h_i)$ is an isomorphism and therefore $H_2(f_i)$ is also an isomorphism. By irreducibility condition I_2, f_i is an homotopy self-equivalence. This completes the proof of Proposition 1. ∎

The irreducibility condition simplifies the verification of cellular equivalences as shown by the following proposition.

PROPOSITION 2: *Let X, Y and T be strictly simply-connected c-finite rational spaces. If $X\vee T$ is an irreducible decomposition relative to T, and $X\vee T \sim Y\vee T$, then there is an H_2-surjective map $k\colon Y \to X$.*

Proof: Since $X \vee T$ and $Y \vee T$ are cellularly equivalent, there are H_2-surjective maps

$$g = \bigvee g_j\colon \bigvee_{j\in J}(Y \vee T) \to X \quad \text{and} \quad f = \bigvee f_i\colon \bigvee_{i\in I}(X \vee T) \to Y.$$

This gives an H_2-surjective map

$$\Big[\bigvee_{j\in J}\bigvee_{i\in I}(g_{j_{|Y}} \circ f_i)\Big] \vee \Big[\bigvee_{j\in J} g_{j_{|T}}\Big]\colon \Big(\bigvee_{j\in J}\Big(\bigvee_{i\in I} X \vee T\Big)\Big) \vee \Big(\bigvee_{j\in J} T\Big) \to X.$$

By Lemma 1, each map $g_{j_{|Y}} \circ f_i$ factors as

$$X \vee T \overset{h_{i,j}}{\to} Y_{i,j} \overset{g_{i,j}}{\to} X,$$

such that each $h_{i,j}$ is H_2-surjective, each $g_{i,j}$ is H_2-injective, and $(\bigvee_{i,j} g_{i,j}) \vee (\bigvee_j g_{j_{|T}})$ is H_2-surjective. Therefore $X \vee T$ is cellularly equivalent to $(\bigvee_{i,j} Y_{i,j}) \vee T$.

Since the space $X \vee T$ is an irreducible decomposition relative to T, there is at least one pair (i_0, j_0) such that $H_2(g_{j_0} \circ f_{i_0|_X})$ is an isomorphism. Thus $H_2(g_{j_0|_Y})$ is a surjective map. The result follows as we set $k = g_{j_0}|_Y$. ∎

The next two propositions are useful to detect irreducible spaces.

PROPOSITION 3: *Let X be a strictly simply-connected c-finite rational space. If the n-skeleton Y of X satisfies the irreducibility property I_1, then X also satisfies I_1.*

Proof: If X is cellularly equivalent to a wedge $\bigvee_{i\in I} X_i$, then Y is cellularly equivalent to the wedge $\bigvee_{i\in I} Y_i$, where Y_i is the rational n-skeleton of X_i. Therefore $\dim H_2(X_i) = \dim H_2(Y_i) \geq \dim H_2(X)$ for at least one i. ∎

PROPOSITION 4: *Let Y be a strictly simply-connected c-finite rational space. If the m-skeleton of Y satisfies irreducibility condition I_2, and the Hurewicz map $h_q\colon \pi_q(Y) \to H_q(Y)$ is zero for $q > m$, then Y also satisfies I_2.*

Proof: It is enough to consider the case where Y is obtained from a space X of dimension m satisfying I_2 by adjunction of n rational cells of dimension $r > m$, i.e.,

$$\left(\bigvee_{i=1}^{n} S_i^{r-1}\right)_0 \xrightarrow{\varphi} X \to Y$$

is a cofibration sequence. Denote by $K =$ the image of $\pi_{r-1}(\varphi)$.

Suppose f is a self-map of Y inducing an isomorphism on H_2. The map f restricts to a homotopy self-equivalence f_X of X. The commutative diagram

$$\begin{array}{ccc} \pi_{r-1}(X) & \longrightarrow & \pi_{r-1}(Y) = \pi_{r-1}(X)/K \\ \downarrow{\scriptstyle \pi_{r-1}(f_X)} & & \downarrow{\scriptstyle \pi_{r-1}(f)} \\ \pi_{r-1}(X) & \longrightarrow & \pi_{r-1}(Y) = \pi_{r-1}(X)/K \end{array}$$

shows that $\pi_{r-1}(f_X)$ maps K isomorphically onto itself. By the hypothesis on the Hurewicz map, $\pi_{r-1}(\varphi)$ is injective. This implies the existence of a rational homotopy equivalence ψ making the diagram

$$\begin{array}{ccccc} (\bigvee_{i=1}^{n} S_i^{r-1})_0 & \xrightarrow{\varphi} & X & \longrightarrow & Y \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle f_X} & & \\ (\bigvee_{i=1}^{n} S_i^{r-1})_0 & \xrightarrow{\varphi} & X & \longrightarrow & Y \end{array}$$

commutative.

When $r = 3$, X is a wedge of rational sphere S_0^2, and since $h_3(Y) = 0$, Y is also a wedge of rational sphere S_0^2, and the result follows in that case.

We now suppose $r > 3$. If $H_{r-1}(\varphi) \neq 0$, then $X \cong Z \vee S^{r-1}$, and X does not satisfy property I_2. Therefore $H_{r-1}(\varphi) = 0$. The naturality of the long exact homology sequence of the cofibration sequence $(\bigvee_{i=1}^{n} S_i^{r-1})_0 \xrightarrow{\varphi} X \to Y$ implies the commutativity of the diagram

$$\begin{array}{ccc} H_r(Y) & \xrightarrow{\cong} & H_{r-1}(\bigvee S_i^{r-1}; \mathbb{Q}) \\ \downarrow{\scriptstyle H_r(Y)} & & \downarrow{\scriptstyle \psi} \\ H_r(Y) & \xrightarrow{\cong} & H_{r-1}(\bigvee S_i^{r-1}; \mathbb{Q}). \end{array}$$

This shows that $H_r(f)$ is an isomorphism, and therefore that f is a homotopy self-equivalence. ∎

An important property of irreducible spaces is the following, very useful proposition.

PROPOSITION 5: *Let X and Y be strictly simply-connected c-finite irreducible rational spaces. Then X is cellularly equivalent to Y if and only if X and Y have the same homotopy type.*

Proof: If $X \sim Y$, by Proposition 2, there are H_2-surjective maps $f: X \to Y$ and $g: Y \to X$. In particular $H_2(g \circ f)$ and $H_2(f \circ g)$ are isomorphisms and by the irreducibility property I_2, f and g are homotopy inverse self-equivalences. ∎

3.2. CONSTRUCTION OF IRREDUCIBLE DECOMPOSITIONS.

PROPOSITION 6: *Let X be a strictly simply-connected c-finite rational space. Then X is cellularly equivalent to a simply-connected c-finite rational space Y satisfying property I_2, and such that $\dim H_2(Y) = \dim H_2(X)$.*

Proof: Let $f: X \to X$ be a continuous map that induces an isomorphism on H_2. We denote by r the least integer n such that $f_{|_{\pi_n(X)}}$ is not injective, and by α an element in $\pi_r(X)$ such that $\pi_r(f)(\alpha) = 0$. Then f factors as $f = f' \circ i$, where

$$X \overset{i}{\hookrightarrow} (X \cup_\alpha e^{r+1})_0 \overset{f'}{\to} X \overset{i}{\to} (X \cup_\alpha e^{r+1})_0.$$

We observe that the space $(X \cup_\alpha e^{r+1})_0$ is cellularly equivalent to X and that

$$\dim \pi_r(X \cup_\alpha e^{r+1})_0 < \dim \pi_r(X).$$

Let N be the cohomological dimension of X. We denote by $\mathcal{C}$ the set of homotopy classes of strictly simply-connected c-finite rational spaces Y cellularly equivalent to X that have cohomological dimension less than or equal to N and satisfy $\dim H_2(Y) = \dim H_2(X)$. For $Y \in \mathcal{C}$, let $n_i(Y) = \dim \pi_i(Y)$. We give the set of sequences

$$(n_3(Y), n_4(Y), \ldots, n_{N-1}(Y))$$

the lexicographic order, and we choose a space Y corresponding to a minimal sequence.

If the Hurewicz map $h_N: \pi_N(Y) \to H_N(Y)$ is nonzero, then Y is homotopy equivalent to $Y' \vee S_0^N$ and we replace Y by Y'. We can therefore suppose that $h_N = 0$.

Let $f: Y \to Y$ be a continuous map inducing an isomorphism on H_2. If, for some $r < N$, $\pi_r(f)$ is not an isomorphism then Y is cellularly equivalent to some space Y' corresponding to a smaller sequence $n_i(Y')$. Therefore f induces an isomorphism on $\pi_{<N}$.

Let $(\wedge Z, d)$ be the Sullivan minimal model of Y, and $\varphi\colon (\wedge Z, d) \to (\wedge Z, d)$ a minimal model for f. By hypothesis φ induces an isomorphism on $(\wedge Z)^{<N} \oplus (\wedge^{\geq 2} Z)^N$. The map φ induces therefore an isomorphism in cohomology and so f is a homotopy self-equivalence. This shows that Y satisfies I_2. ∎

We now proceed to the proof of the decomposition theorems. We first note that if X is cellularly equivalent to $\bigvee_{i\in I} X_i$ then there is an H_2-surjective map $\bigvee_k (\bigvee_{i\in I} X_i) \to X$. This directly implies that there is a subset $J \subset I$ such that $X \sim \bigvee_{j\in J} X_j$ with $|J| \leq \dim H_2(X)$.

THEOREM 3: *Let X and Y be strictly simply-connected c-finite rational spaces. Then $X \vee Y$ admits an irreducible decomposition relative to Y.*

Proof: If the space X satisfies the irreducibility property I_1, then by Proposition 6, we have a cellular equivalence $X \sim Z$ with Z irreducible. If the space X does not satisfy I_1, then $X \sim \bigvee_{i\in I} X_i$, where by hypothesis, for each i, $\dim H_2(X_i) < \dim H_2(X)$. We can also assume that the cardinality of I is less than or equal to the dimension of $H_2(X)$, and that none of the X_i is built by the other ones. By iterating the decomposition process we can suppose that the X_i satisfy property I_1, and by Proposition 4, property I_2.

We now consider all the decompositions $X \vee Y \sim \bigvee_{i\in I} X_i \vee Y$ such that

1. each X_i is irreducible;
2. card $I < \dim H_2(X)$;
3. $\dim H_2(X_i) < \dim H_2(X)$ for each $i \in I$; and
4. no X_i is built by the space Y and the other X_j.

To such a decomposition we associate a sequence

$$(m_1, m_2, \ldots, m_q)$$

where $q = \dim H_2(X)$, and m_j is the number of components X_i with $\dim H_2(X_i) = j$.

Since card $I \leq \dim H_2(X)$, $\sum m_i \leq q$. We give the set of such sequences the lexicographic order, and we choose a decomposition corresponding to a maximal sequence, i.e.,

$$X \vee Y \sim \bigvee_{i\in I} X_i \vee Y.$$

The chosen decomposition is irreducible: Suppose that for some $i_0 \in I$, we have an H_2-surjective map

$$f = \bigvee_k f_k\colon \bigvee_{k\in K} \left(\bigvee_{i\in I} X_i \vee Y \right) \to X_{i_0}.$$

Each $f_{k|_{X_i}}$ decomposes into $g_{k,i} \circ h_{k,i}$ where $g_{k,i}\colon T_{k,i} \to X_{i_0}$ is an H_2-injective map and $h_{k,i}\colon X_i \to T_{k,i}$ is an H_2-surjective map. Each $f_{k|_Y}$ decomposes into $l_k \circ m_k$ where $l_k\colon Z_k \to X_{i_0}$ is an H_2-injective map and $m_k\colon Y \to Z_k$ is an H_2-surjective map. Moreover, $(\bigvee_{k,i} g_{k,i}) \vee (\bigvee_k l_k)$ is an H_2-surjective map.

Suppose that there does not exist an integer k such that $f_{k|_{X_{i_0}}}$ is H_2-surjective. Since X_{i_0} is not built by Y and the other X_j, no $g_{k,i}$ and no l_k is H_2-surjective. This means that $\dim H_2(T_{k,i}) < \dim H_2(X_{i_0})$ for each pair (k,i), and $\dim H_2(Z_k) < \dim H_2(X_{i_0})$. We deduce a cellular equivalence

$$X \vee Y \sim \Big(\bigvee_{i \neq i_0} X_i\Big) \vee \Big(\bigvee_k T_{k,i_0}\Big) \vee \Big(\bigvee_k Z_k\Big) \vee Y.$$

We decompose the T_{k,i_0} and the Z_k into irreducible elements and we suppress the components that are built by the other factors. This new decomposition corresponds to a sequence that is strictly larger than the previous one, which is impossible. ∎

THEOREM 4 (Unique decomposition theorem): *A strictly simply-connected c-finite rational space admits a unique (up to permutation of the factors) irreducible decomposition.*

Proof: Suppose that X admits two irreducible decompositions

$$X \sim \bigvee_{i \in I} Y_i \sim \bigvee_{j \in J} Z_j.$$

The cellular equivalence implies the existence, for each i, of H_2-surjective maps f and g,

$$\bigvee_{l \in L} \Big(\bigvee_{i \in I} Y_i\Big) \xrightarrow{g = \vee_{l,i} g_{l,i}} \bigvee_{k \in K} \Big(\bigvee_{j \in J} Z_j\Big) \xrightarrow{f = \vee_{k,j} f_{k,j}} Y_i.$$

By the irreducibility of the decomposition $\bigvee_{i \in I} Y_i$ of X, there exists l_0 such that the composite $f \circ g_{l_0,i}\colon Y_i \to Y_i$ is H_2-surjective. Therefore, the map

$$\bigvee_{k,j} (f_{k,j} \circ q_{k,j} \circ g_{l_0,i})\colon \bigvee_{k \in K} \bigvee_{j \in J} Y_i \to Y_i$$

is also H_2-surjective. Here $q_{k,j}$ denotes the projection map

$$\bigvee_k \Big(\bigvee_j Z_j\Big)_k \to \Big(\bigvee_j Z_j\Big)_k \to Z_j.$$

By irreducibility, this implies that for some (k_0, j_0) the map $f_{k_0,j_0} \circ q_{k_0,j_0} \circ g_{l_0,i}$ is H_2-surjective. Therefore f_{k_0,j_0} is H_2-surjective. Let $r(i) = j_0$. We have the inequalities

$$Z_{r(i)} \ll Y_i \quad \text{and}$$
$$\bigvee_{i\in I} Z_{r(i)} \ll \bigvee_{i\in I} Y_i \ll \bigvee_{j\in J} Z_j,$$

and therefore

$$\bigvee_{i\in I} Z_{r(i)} \sim \bigvee_{j\in J} Z_j.$$

Since $Z_{r(i)}$ could be equal to $Z_{r(j)}$ for $i \neq j$, the cardinality of J is less than or equal to the cardinality of I.

A similar argument shows that for each $j \in J$ there is an element $s(j) \in I$ such that $Y_{s(j)} \ll Z_j$. This shows that $|I| = |J|$. In particular, the components $Z_{r(i)}$ are all different, as are all the components $Y_{s(j)}$. Since no Z_j is built by other Z_k and no X_i is built by other X_l, the applications $r\colon I \to J$ and $s\colon J \to I$ are inverse bijections, and

$$Y_i \sim Z_{r(i)}.$$

This proves the theorem. ∎

The determination of the irreducible factors X_i in an irreducible decomposition $X \sim \bigvee_{i\in I} X_i$ reduces in fact to the search of the retracts of X.

PROPOSITION 7: *If $X \sim \bigvee_{i\in I} X_i$ is an irreducible decomposition of a strictly simply-connected c-finite rational space, then each X_i is a homotopy retract of X.*

Proof: Since $X \sim \bigvee_{i\in I} X_i$, there are H_2-surjective maps

$$f = \bigvee_k f_k\colon \bigvee_k X \to \bigvee_{i\in I} X_i \quad \text{and} \quad g = \bigvee g_l\colon \bigvee_l (\bigvee_{i\in I} X_i) \to X.$$

Therefore, for $i_0 \in I$, the composition map

$$\bigvee_k (f \circ g)\colon \bigvee_k \bigvee_l \left(\bigvee_{i\in I} X_i \right) \to \bigvee_{i\in I} X_i \to X_{i_0}$$

is H_2-surjective. By property P_2, there are k and l such that $f_k \circ g_l|_{X_{i_0}}\colon X_{i_0} \to X_{i_0}$ is a homotopy equivalence. This shows that X_{i_0} is a homotopy retract of X. ∎

COROLLARY 1: *Let X be a strictly simply-connected c-finite rational space such that each self-map of X is either the trivial map or a homotopy self-equivalence, then X is irreducible.*

3.3. THE POSET OF GRADED IDEALS IN A FREE LIE ALGEBRA. The purpose of this section is to prove the following theorem.

THEOREM 5: *Let L be the free graded Lie algebra on two generators of the same positive even degree, and $\mathcal{L}$ be the poset of finitely generated graded ideals in L. Then there is an injection θ: $(\mathcal{L}, \subset) \to (\mathcal{B}_0, \ll)$ satisfying $\theta(I) \ll \theta(J)$ if and only if $I \subset J$.*

Proof: Let X be the rationalization of the space

$$[P^4(\mathbb{C})_1 \# P^4(\mathbb{C})_2 \# P^4(\mathbb{C})_3] \cup_{P^2(\mathbb{C})_1 \vee S^2_2} (P^2(\mathbb{C})_1 \times S^2_2),$$

i.e., the rationalization of the pushout of the diagram

$$\begin{array}{ccc} P^2(\mathbb{C})_1 \vee S^2_2 & \xrightarrow{\ i\ } & P^2(\mathbb{C})_1 \times S^2_2 \\ \downarrow{\scriptstyle j} & & \\ P^4(\mathbb{C})_1 \# P^4(\mathbb{C})_2 \# P^4(\mathbb{C})_3 & & \end{array}$$

where i and j denote canonical injections. Since $H^*(i;\mathbb{Q})$ is surjective, the Mayer–Vietoris exact sequence of the pushout yields the isomorphism of algebras

$$\begin{gathered} H^*(X;\mathbb{Q}) = \operatorname{Ker} H^*(i) - H^*(j): \\ H^*(P^2(\mathbb{C})_1 \times S^2_2) \oplus H^*(P^4(\mathbb{C})_1 \# P^4(\mathbb{C})_2 \# P^4(\mathbb{C})_3) \to H^*(P^2(\mathbb{C})_1 \vee S^2_2) \\ = \bigwedge(x_1, x_2, x_3)/(x_1^4 - x_2^4, x_1^4 - x_3^4, x_1x_3, x_2x_3, x_1x_2^2, x_1^3x_2). \end{gathered}$$

LEMMA 2: *The space X is rationally formal.*

Proof: Recall that a space X is rationally formal if its minimal model is quasi-isomorphic to the differential graded algebra $(H^*(X;\mathbb{Q}), 0)$ ([21], [10], page 156), or equivalently if X admits a Quillen minimal model, $(\mathbb{L}(W), d)$ with a purely quadratic differential, $d(W) \subset \mathbb{L}^2(W)$ ([22]). Since $W_q = H_{q+1}(X;\mathbb{Q})$ ([10], formula 24.3), if i: $S \to T$ is the injection of a subcomplex and if $H_*(i;\mathbb{Q})$ is injective, then the Quillen minimal model of i, φ_i: $(\mathbb{L}(W_S), d) \to (\mathbb{L}(W_T), d)$ is injective. Therefore, if Y is the union of the formal subcomplexes Y_α, and if each inclusion $Y_\alpha \to Y$ induces an injective map in rational homology, then Y is also formal. This applies directly to the space X. ∎

Denote by $\alpha\colon S^6 \to \Omega P^3(\mathbb{C})$ the usual Hopf map, and by $\alpha_1 \in \pi_6(\Omega P^3(\mathbb{C})_1)$ and $\alpha_2 \in \pi_6(\Omega P^3(\mathbb{C})_2)$ the corresponding elements in $P^3(\mathbb{C})_1$ and $P^3(\mathbb{C})_2$. Then

LEMMA 3: *The elements α_1 and α_2 generate a free Lie subalgebra in $\pi_*(\Omega X)\otimes\mathbb{Q}$.*

Proof: Since X is formal, its minimal model, $(\wedge Z, d)$, is the minimal model of $(H^*(X;\mathbb{Q}),0)$,

$$\varphi\colon (\wedge Z, d) \overset{\cong}{\to} (H^*(X;\mathbb{Q}),0).$$

By ([13]), this minimal model admits a second gradation, $Z = \bigoplus_{p\geq 0} Z_p$, such that $d(Z_p) \subset (\wedge Z)_{p-1}$,

$$\begin{gathered} Z_0 = Z_0^2 = (x_1,x_2,x_3), \quad \varphi(x_1)=x_1, \quad \varphi(x_2)=x_2, \quad \varphi(x_3)=x_3,\\ Z_1 = (y_1,y_2,y_3,z_1,z_2,z_3), \quad dy_1 = x_1x_3, \quad dy_2 = x_2x_3, \quad dy_3 = x_1x_2^2,\\ dz_1 = x_1^4 - x_3^4, \quad dz_2 = x_2^4 - x_3^4, \quad dz_3 = x_1^3x_2,\\ |y_1| = |y_2| = 3, \quad |y_3| = 5, \quad |z_1| = |z_2| = |z_3| = 7,\\ \varphi(Z_{\geq 1}) = 0. \end{gathered}$$

The model of the injection $k\colon P^3(\mathbb{C})_1 \to X$ is given by

$$\begin{gathered} K\colon (\wedge Z, d) \to (\wedge(a_1,a_2),d), \quad |a_1| = 2, \quad |a_2| = 7, \quad da_2 = a_1^4,\\ K(x_1) = a_1, \quad K(z_1) = a_2,\\ K(x_2) = K(x_3) = K(z_2) = K(z_3) = K(y_i) = K(Z_{\geq 1}) = 0. \end{gathered}$$

Let $\tilde\alpha \in \pi_7(P^3(\mathbb{C}))$ and $\tilde\alpha_1$ and $\tilde\alpha_2 \in \pi_7(X)$ be the elements corresponding by adjunction to α, α_1 and α_2.

Recall that the minimal model $(\wedge R, d)$ of a simply-connected finite-type CW-complex S is equipped with a natural isomorphism $R^n \cong \mathrm{Hom}(\pi_n(S),\mathbb{Q})$. Here $a_2\colon \pi_7(P^3(\mathbb{C})) \to \mathbb{Q}$ satisfies $\langle a_2;\tilde\alpha\rangle = 1$. By naturality, we have

$$\langle z_1;\tilde\alpha_1\rangle = \langle z_1;\pi_7(k)(\tilde\alpha)\rangle = \langle K(z_1),\tilde\alpha\rangle = \langle a_2;\tilde\alpha\rangle = 1.$$

Similarly, $\langle z_2,\tilde\alpha_2\rangle = 1$.

By ([10], 15.c), the minimal model of the 2-connected cover of X, $X[2]$, is the quotient differential graded algebra

$$(\wedge Z/(x_1,x_2,x_3),\bar d).$$

The quotient map

$$\begin{gathered} \psi\colon (\wedge Z \otimes \wedge(r,s,t), D) \to (\wedge Z \otimes \wedge(r,s,t)/(x_1,x_2,x_3,r,s,t), \overline{D}),\\ |r| = |s| = |t| = 1, \quad dr = x_1, \quad ds = x_2, \quad dt = x_3, \end{gathered}$$

is a quasi-isomorphism of differential graded algebras. On the other hand, the quasi-isomorphism $\varphi\colon (\wedge Z, d) \overset{\cong}{\to} (H^*(X;\mathbb{Q}), 0)$ extends to the quasi-isomorphism

$$\varphi \otimes 1\colon (\wedge Z \otimes \wedge(r,s,t), D) \to (H^*(X;\mathbb{Q}) \otimes \wedge(r,s,t), D).$$

Denote by θ a homotopy inverse of ψ. For degree reasons, we have

$$\theta(z_1) = z_1 - rx_1^3 + tx_3^3, \quad \theta(z_2) = z_2 - sx_2^3 + tx_3^3.$$
$$((\varphi \otimes 1) \circ \theta)(z_1) = -rx_1^3 + tx_3^3, \quad ((\varphi \otimes 1) \circ \theta)(z_2) = -sx_2^3 + tx_3^3.$$

This shows that $((\varphi \otimes 1) \circ \theta)(z_1) \cdot ((\varphi \otimes 1) \circ \theta)(z_2) = 0$, and implies the existence of the morphism ρ in the following diagram, $\rho(u) = ((\varphi \otimes 1) \circ \theta)(z_1)$, $\rho(v) = ((\varphi \otimes 1) \circ \theta)(z_2)$.

$$\begin{array}{ccc}
(\wedge T, d) & \xrightarrow{\quad \tilde{\rho} \quad} & (\wedge Z/(x_1,x_2,x_3), d) \\
\downarrow{\scriptstyle \varepsilon} & & \simeq \downarrow{\scriptstyle (\varphi \otimes 1)\circ\theta} \\
(\wedge(u,v)/(uv), 0) & \xrightarrow{\ \rho\ } & (H^*(X;\mathbb{Q}) \otimes \wedge(r,s,t), D).
\end{array}$$

Here ε is the minimal model of $(\wedge(u,v)/(uv), 0)$, $(\wedge T, d) = (\wedge(a,b,c,\ldots), d)$, with $\varepsilon(a) = u$, $\varepsilon(b) = v$ and $d(c) = ab$. The differential graded algebra $(\wedge T, d)$ is the minimal model of $S^7 \vee S^7$. The morphism $\tilde{\rho}$ follows from the Lifting Lemma ([10] Proposition 12.9): $(\varphi \otimes 1) \circ \theta \circ \tilde{\rho} \sim \rho \circ \varepsilon$. The construction of $\tilde{\rho}$ can be realized by induction on the degrees of the generators. We can therefore suppose that $\tilde{\rho}(a) = z_1$ and $\tilde{\rho}(b) = z_2$.

The geometric realization functor ([21], [10], Section 17) transforms $\tilde{\rho}$ into a continuous map $\bar{\rho}\colon Z[2]_0 \to (S^7 \vee S^7)_0$. Denote by $\tilde{\omega}_1$ and $\tilde{\omega}_2$ generators of $\pi_7(S^7 \vee S^7) \otimes \mathbb{Q}$ such that $\langle a, \tilde{\omega}_1 \rangle = 1$ and $\langle b, \tilde{\omega}_2 \rangle = 1$. Then, $\langle a, \pi_7(\bar{\rho})(\tilde{\alpha}_1) \rangle = \langle \tilde{\rho}(a), \tilde{\alpha}_1 \rangle = 1$. Therefore, $\pi_7(\bar{\rho})(\tilde{\alpha}_1) = \tilde{\omega}_1$. In the same way, $\pi_7(\bar{\rho})(\tilde{\alpha}_2) = \tilde{\omega}_2$.

Recall finally that $\pi_*(\Omega(S^7 \vee S^7)) \otimes \mathbb{Q}$ is a free Lie algebra on the generators ω_1 and ω_2 corresponding by adjunction to $\tilde{\omega}_1$ and $\tilde{\omega}_2$. Since $\pi_*(\Omega\bar{\rho}) \otimes \mathbb{Q}$ is a morphism of graded Lie algebras, the elements α_1 and α_2 generate a free Lie subalgebra in $\pi_*(\Omega X) \otimes \mathbb{Q}$. ∎

Let $\varphi\colon X \to X$ be a self-map. Denote by $\tilde{\varphi}$ the induced automorphism of the minimal model $(\wedge Z, d)$ of X. With the notations of the proof of Lemma 3, a computation shows that, for some $a \in \mathbb{Q}$, we have

$$\tilde{\varphi}(x_1) = ax_1, \quad \tilde{\varphi}(x_2) = \pm ax_2 \quad \text{and} \quad \tilde{\varphi}(x_3) = \pm ax_3.$$

We then deduce that $\pi_*(\Omega\varphi)(\alpha_1) = a^4\alpha_1$ and $\pi_*(\Omega\varphi)(\alpha_2) = a^4\alpha_2$. Therefore, the graded ideals of the free Lie algebra F on α_1 and α_2 are preserved by the homotopy self-equivalences of X. We denote by L the sub-Lie algebra generated by $[\alpha_1,[\alpha_1,\alpha_2]]$ and $[\alpha_2,[\alpha_1,\alpha_2]]$.

Let I be a finitely generated graded ideal in L. Choose a minimal system of generators $\omega_1,\dots,\omega_n$ of I such that each ω_i is a graded homogeneous element of I. We define the space

$$X_I = X \cup_{\tilde{\omega}_1} e^{|\omega_1|+2} \cup \cdots \cup_{\tilde{\omega}_n} e^{|\omega_n|+2}.$$

Since each self-map of X is either the trivial map or a homotopy self-equivalence, by Corollary 1, X is irreducible and thus X_I is also irreducible.

Let $\mathcal{L}$ be the poset of finitely generated ideals in L; then the correspondance

$$I \to X_I$$

induces a morphism of posets

$$\theta\colon (\mathcal{L},\subset) \to (\mathcal{B}_0,\ll).$$

By Proposition 5, this map θ is injective. ∎

4. Constructions of strict inequalities in $\mathcal{B}$ and $\mathcal{B}_0$

In this section we introduce tools to construct strict cellular inequalities.

Proposition 8: *Let $X\vee T$ be an irreducible decomposition relative to T with X a simply-connected irreducible m-dimensional finite CW-complex Bousfield equivalent to the sphere S^2. Let $\alpha\in\pi_q(X)$ such that $q>m$ and $\alpha\otimes 1\neq 0$ in $\pi_q(X)\otimes\mathbb{Q}$. Then we have strict inequalities*

$$X\vee T \overset{s}{\ll} (X\cup_\alpha e^{q+1})\vee T \overset{s}{\ll} T \quad\text{and}\quad X_0\vee T_0 \overset{s}{\ll} (X\cup_\alpha e^{q+1})_0\vee T_0 \overset{s}{\ll} T_0.$$

Proof: Since X is Bousfield equivalent to S^2, ΣX builds S^3, and therefore X builds any space of the form $X\cup_\omega e^n$ with $n\geq 4$. The cellular injection $X\hookrightarrow Y = X\cup_\alpha e^{q+1}$ shows that $X\ll Y$. Suppose the spaces $X\vee T$ and $Y\vee T$ are cellularly equivalent. The decomposition of $X\vee T$ being irreducible, by Proposition 2, there is an H_2-surjective map $g\colon Y_0\to X_0$. Since X is the m-skeleton of Y, and the space X is irreducible, g restricts to an H_2-surjective map $g_X\colon X_0\to X_0$. Moreover, X being irreducible, g_X is an homotopy equivalence. Denote by $i_0\colon X_0\to Y_0$ the canonical injection. Then the commutativity of the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{g_X} & X_0 \\ {\scriptstyle i_0}\downarrow & & \| \\ Y_0 & \xrightarrow{g} & X_0 \end{array}$$

implies the injectivity of $\pi_q(i_0)$, which is impossible because $\alpha \oslash 1 \neq 0$. ∎

PROPOSITION 9: *Let X be a strictly simply-connected irreducible m-dimensional finite CW-complex. Let $\alpha \in \pi_q(X)$ and $\beta \in \pi_r(X)$ such that*

1. $[\alpha] = 0$ *in* $\pi_q(X \cup_\beta e^{r+1})$;
2. $\alpha \oslash 1$ *and* $\beta \oslash 1$ *are nonzero in* $\pi_*(X) \oslash \mathbb{Q}$; *and*
3. $m < r < q$.

Then, we have strict inequalities

$$Y = X \cup_\alpha e^{q+1} \overset{s}{\ll} Z = X \cup_\beta e^{r+1} \quad \text{and} \quad Y_0 = (X \cup_\alpha e^{q+1})_0 \overset{s}{\ll} Z_0 = (X \cup_\beta e^{r+1})_0.$$

Proof: The class $[\alpha]$ being trivial in $\pi_q(X \cup_\beta e^{r+1})$, we have a homotopy equivalence

$$Y \cup_\beta e^{r+1} \simeq Z \vee S^{q+1}.$$

Therefore Y builds Z. Moreover, the inequality is strict, and even strict rationally. Indeed, suppose that $Z_0 \ll Y_0$. Since X is irreducible, the same is true for Y and Z. Thus there exists an H_2-surjective map $f\colon Z_0 \to Y_0$. Since X is the m-skeleton of Y and Z, and X is irreducible, the restriction f_X of f to X_0 is a homotopy self-equivalence.

Consider the following commutative diagram in which the vertical maps are injections of skeleta:

$$\begin{array}{ccc} X_0 & \xrightarrow{f_X} & X_0 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ Z_0 & \xrightarrow{f} & Y_0. \end{array}$$

The homomorphism $\pi_s(j)$ being injective for $s \leq q-1$, the map $\pi_s(i)$ is also injective for $s \leq q-1$, but this is impossible since $r < q$ and $\pi_r(i)(\beta) = 0$. ∎

5. The space R_Z for an irreducible decomposition $Z \vee T$ with Z elliptic

In this section all spaces will be strictly simply-connected c-finite rational spaces. Our starting point is an irreducible decomposition $Z \vee T$, relative to T, with Z elliptic. Our goal is to construct another irreducible decomposition relative to T, $R_Z \vee T$ with R_Z hyperbolic, and

$$Z \vee T \overset{s}{\ll} R_Z \vee T \overset{s}{\ll} T.$$

We fix first some notation: Let n be the least integer m such that $\pi_{\geq 2m}(Z) = 0$, α a non-trivial element in $\pi_{2n-1}(Z)$, N the cohomological dimension of $Z \vee T$, and M an integer greater than $2 \cdot \dim H_n(Z \vee T)$. Recall from the introduction the construction of R_Z:

$$\text{when } n = 2, \quad R_Z = \left(\left[Z \times \left(\bigvee_{i=1}^{2M} P_i^N(\mathbb{C})\right)\right] \cup_{\alpha+\omega} e^4\right)_0, \quad \omega = \sum_{i=1}^{M}[\beta_{2i-1}, \beta_{2i}]$$

and

$$\text{when } n > 2, \quad R_Z = \left(\left[Z \times \left(\bigvee_{i=1}^{2M} S_i^n\right)\right] \cup_{\alpha+\omega} e^{2n}\right)_0, \quad \omega = \sum_{i=1}^{M}[\beta_{2i-1}, \beta_{2i}],$$

where the spaces $P_i^N(\mathbb{C})$ are different copies of the projective space $P^N(\mathbb{C})$, the spaces S_i^n are different copies of the sphere S^n, and the β_i are generators of $\pi_2(P_i^N(\mathbb{C}))$ or $\pi_n(S_i^n)$.

Throughout section 3, let X_i be either $(P_i^N(\mathbb{C}))_0$ or $(S_i^n)_0$ according to whether $n = 2$ or $n > 2$. Moreover, in what follows all rational spaces are supposed to be equipped with a rational cell decomposition such that the suspensions of the attaching maps are trivial. We denote by T^n the n-th skeleton of the space T. In the same way all continuous maps $f\colon T \to V$ are supposed to be cellular, i.e., for any n, $f(T^n) \subset V^n$. The properties of R_Z are contained in Propositions 10 to 13.

LEMMA 4: *The image of α by the Hurewicz map $h_{2n-1}\colon \pi_{2n-1}(Z) \to H_{2n-1}(Z)$ is zero.*

Proof: Let $(\bigwedge V, d)$ be the Sullivan minimal model of the space Z. Since $V^r = \operatorname{Hom}(\pi_r(Z), \mathbb{Q})$, $V^{>2n-1} = 0$. The Hurewicz map h_{2n-1} is dual to the map induced in homology by the projection of complexes

$$\bar{h}\colon (\textstyle\bigwedge^+ V, d) \to (\bigwedge^+ V/\bigwedge^{\geq 2} V, \bar{d}) \cong (V, 0).$$

If $h_{2n-1} \neq 0$, then $\bar{h}_{2n-1} \neq 0$ and $(\bigwedge V, d)$ contains a cocycle a of the form $a = \sum_{i\geq 1} z_i$, with $z_i \in \bigwedge^i V$ and $z_1 \neq 0$. Denote by V' a graded complement of $\mathbb{Q}z_1$ in V. For degree reasons, $d(V') \subset \bigwedge^{\geq 2} V'$. We have therefore an isomorphism of differential graded algebras

$$(\textstyle\bigwedge V, d) \cong (\bigwedge V', d) \odot (\bigwedge a, 0).$$

This implies that $Z \cong Z' \times S^{2n-1}$, which is not an irreducible space. ∎

Lemma 4 implies that the homology class $[e^{2n}]$, generated by the cell e^{2n} in the homology of the cellular chain complex of R_Z, is non-zero.

LEMMA 5: *We have inequalities*

$$Z \vee T \ll R_Z \vee T \ll T.$$

Proof: The only nontrivial case is the case $n = 2$. Since N is the cohomological dimension of $Z \vee T$, there exist H_2-surjective maps $Z \to P^N(\mathbb{C})$, and this shows that $Z \ll R_Z$. ∎

PROPOSITION 10: $Z \vee T \overset{s}{\ll} R_Z \vee T$.

Proof: Suppose there exists an H_2-surjective map

$$\varphi = \bigvee_k \varphi_k \colon \bigvee_{k \in K} (R_Z \vee T) \to Z.$$

When $n = 2$, the restrictions $\varphi|_{P_i^N(\mathbb{C})} \colon P_i^N(\mathbb{C}) \to Z$ induce the zero map on H_2 by the choice of the integer N. Therefore, for all values of n, the map φ induces an H_2-surjective map

$$\varphi' = \bigvee_k (\varphi_k|_{Z \vee T}) \colon \bigvee_k (Z \vee T) \to Z.$$

By the irreducibility hypothesis on $Z \vee T$, there is an index k_0 for which the restriction $\varphi_{k_0}|_Z \colon Z \to Z$ is an H_2-isomorphism. Since Z is irreducible, $\varphi_{k_0}|_Z$ is a homotopy equivalence. We denote by ψ a homotopy inverse. We can therefore suppose that $\psi \circ \varphi_{k_0} \colon R_Z \to Z$ restricts to the identity on Z.

We denote by θ the restriction of $\psi \circ \varphi_{k_0}$ to $Z \times (\bigvee_{i=1}^{2M} X_i)$. The map

$$\theta \colon Z \times \left(\bigvee_{i=1}^{2M} X_i \right) \to Z$$

extends to R_Z and is the identity on Z. Therefore,

$$0 = \pi_*(\theta)(\alpha + \omega) = \alpha + \pi_*(\theta)(\omega).$$

Let $aut_1 Z$, as usual, be the (strictly associative) monoid formed by the homotopy self-equivalences of Z, and $ev \colon aut_1 Z \to Z$ the evaluation map at the base point. The map θ induces a map $\tilde{\theta}$ making the following diagram

$$\begin{array}{ccccc} S^{2n-1} & = & = & = & S^{2n-1} \\ \downarrow{\scriptstyle \omega} & & & & \downarrow{\scriptstyle \alpha} \\ (\bigvee_{i=1}^{2M} X_i)_0 & \xrightarrow{\tilde{\theta}} & aut_1 Z & \xrightarrow{ev} & Z \end{array}$$

commutative up to homotopy.

By the universal property of the James functor J (see, e.g., [23], Theorem VII, 2.5), $\tilde{\theta}$ factors through $J(\bigvee_{i=1}^{2M} X_i)$, i.e.,

$$\bigvee_{i=1}^{2M} X_i \xrightarrow{i} J\left(\bigvee_{i=1}^{2M} X_i\right) \xrightarrow{\theta'} aut_1 Z.$$

Recall that $J(S) \cong \Omega\Sigma S$, and that the Hurewicz map $h_{\Omega E}\colon \pi_*(\Omega E) \to H_*(\Omega E)$ is always injective for finite rational spaces ([10], Theorem 16.10). We then consider the following commutative diagram:

$$\begin{array}{ccc}
\pi_{2n-1}(\bigvee_{i=1}^{2M} X_i) & \xrightarrow{h_{\vee_i X_i}} & H_{2n-1}(\bigvee_{i=1}^{2M} X_i;\mathbb{Q}) \\
\downarrow{\scriptstyle \pi_{2n-1}(i)} & & \downarrow{\scriptstyle H_{2n-1}(i)} \\
\pi_{2n-1}(J(\bigvee_{=1}^{2M} X_i)) & \xrightarrow{h_{J(\vee_i X_i)}} & H_{2n-1}(J(\bigvee_{i=1}^{2M} X_i);\mathbb{Q}) \\
\downarrow{\scriptstyle \pi_{2n-1}(\theta')} & & \\
\pi_{2n-1}(aut_1 Z) & & \\
\downarrow{\scriptstyle \pi_{2n-1}(ev)} & & \\
\pi_{2n-1}(Z). & &
\end{array}$$

Since $\pi_{2n-1}(ev \circ \theta' \circ i)([\omega]) = -[\alpha] \neq 0$, we have $\pi_{2n-1}(i)([\omega]) \neq 0$, and thus $h_{J(\vee X_i)} \circ \pi_{2n-1}(i)([\omega]) \neq 0$. Therefore $h_{\vee_i X_i}([\omega]) \neq 0$, which is impossible. ∎

Let $\rho\colon R_Z \to R' = ((\bigvee_{i=1}^{2M} X_i) \cup_\omega e^{2n})_0$ be the canonical projection obtained by mapping the subspace Z to the base point.

The cohomology of R' is given by

$$H^*(R') = \mathbb{Q}[x_1, \ldots, x_{2M}]/I, \quad |x_i| = n,$$

where I is the ideal generated by the elements

$$\begin{cases} x_{2i-1}x_{2i} - x_1x_2, \quad i = 2, \ldots, M; \\ x_ix_j \quad \text{if } |i-j| \geq 2; \text{ and} \\ x_i^p, i = 1, \ldots, 2M, \end{cases}$$

with $p = 2$ if $n > 2$, and $p = N + 1$ if $n = 2$.

The projection ρ induces an injection in rational cohomology. Let Ω' be the cohomology class given by $x_1x_2 \in H^{2n}(R')$, and Ω its image in $H^{2n}(R_Z)$.

LEMMA 6: *Let $g\colon S \to R'$ be a continuous map. Suppose that* $\dim H_n(S) < M$; *then* $H^{2n}(g)(\Omega') = 0$.

Proof: There exists, by hypothesis, a nonzero element $\sum_{i=1}^{M} \alpha_i x_{2i-1} \in H^n(R')$, with $\alpha_i \in \mathbb{Q}$ such that $H^n(g)(\sum_i \alpha_i x_{2i-1}) = 0$. Choose i_0 with $\alpha_{i_0} \neq 0$. In this case,

$$0 = H^{2n}(g)\left(\sum_{i=1}^{M} \alpha_i x_{2i-1} \cdot x_{2i_0}\right) = H^{2n}(g)(\alpha_{i_0} x_{2i_0-1} x_{2i_0}) = \alpha_{i_0} H^{2n}(g)(\Omega),$$

which implies that $H^{2n}(g)(\Omega) = 0$. ∎

PROPOSITION 11: $R_Z \vee T \overset{s}{\ll} T$.

Proof: Suppose we have an H_2-surjective map

$$f\colon \bigvee_{k\in K} T \to R_Z.$$

By the cellular approximation theorem, f is homotopic to a map g that maps $(\bigvee_{k\in K} T)^{2n-1}$ into $Z \times (\bigvee_{i=1}^{2M} X_i)$. We have therefore a commutative diagram

$$\begin{array}{ccccc} (\bigvee_{k\in K} T)^{2n-1} & \xrightarrow{\quad g \quad} & Z \times (\bigvee_{i=1}^{2M} X_i) & \xrightarrow{\quad q \quad} & Z \\ \downarrow & & \downarrow i & & \\ (\bigvee_{k\in K} T)^{2n} & \xrightarrow{\ g\ } & ((Z \times (\bigvee_{i=1}^{2M} X_i)) \cup_{\alpha+\omega} e^{2n})_0, & & \end{array}$$

in which the vertical lines are canonical injections, and q is the standard projection on the first factor.

The obstruction to extending $g|_{(\bigvee_k T)^{2n-1}}$ to a morphism $g'\colon (\bigvee_{k\in K} T)^{2n} \to Z \times (\bigvee_i X_i)$ is a linear map

$$ob(g)\colon H_{2n}\left(\bigvee_{k\in K} T\right) \to \pi_{2n-1}(Z) \oplus \pi_{2n-1}\left(\bigvee_i X_i\right).$$

If the obstruction is zero, then $q \circ g'$ extends to a map $\bar{g}\colon \bigvee_{k\in K} T \to Z$, since $\pi_{\geq 2n}(Z) = 0$. The homomorphism $H_2(g)$ being surjective, $H_2(\bar{g})$ is also surjective, and this implies that $T \ll Z$, which is impossible by Proposition 10 and Lemma 4.

The obstruction map $ob(g)$ is thus nonzero. There is therefore some $k \in K$, with corresponding space T denoted by T_k, such that $ob(g)\colon H_{2n}(T_k) \to$

$\pi_{2n-1}(Z) \oplus \pi_{2n-1}(\bigvee_i X_i)$ is nonzero. Since the image of $ob(g)$ is contained in the kernel of $\pi_{2n-1}(i) = \mathbb{Q}(\alpha + \omega)$, there exists a $(2n)$-cell in T_k with attaching map φ: $S_0^{2n-1} \to (T_k)^{2n-1}$. This makes the following diagram commutative,

$$\begin{array}{ccc}
S_0^{2n-1} & = & S_0^{2n-1} \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle g\circ\varphi} \\
(T_k)^{2n-1} & \xrightarrow{\ g\ } & Z \times (\bigvee_{i=1}^{2M} X_i) \\
\downarrow & & \downarrow \\
((T_k)^{2n-1} \cup_\varphi e^{2n})_0 & \xrightarrow{\ g\ } & R_Z,
\end{array}$$

where the vertical lines are cofibration sequences. Since the suspension of the attaching map φ is trivial, $H_{2n-1}(\varphi) = 0$. The naturality of the homology long exact sequence of a cofibration shows that $[e^{2n}]$ belongs to the image of $H_{2n}(g)$. Therefore $H^{2n}(g)(\Omega) \neq 0$. By composition with the projection ρ: $R_Z \to R'$ we obtain a map $\rho \circ g$: $((T_k)^{2n-1} \cup_\varphi e^{2n})_0 \to R'$ such that $H^{2n}(\rho \circ g)(\Omega') \neq 0$, which is impossible by Lemma 6. ∎

Proposition 12: *When $M \geq 2$, the space R_Z is hyperbolic.*

Proof: The space $L = \bigvee_{i=1}^{M} X_{2i}$ is clearly a retract of R_Z. Since the cohomology of L does not satisfy Poincar duality, L is a hyperbolic space, and thus its rational homotopy grows exponentially. Since L is a retract of R_Z the same is true for R_Z, and therefore R_Z is hyperbolic. ∎

Lemma 7: *Let f: $R_Z \vee T \to R_Z$ be a continuous map. Then f is homotopic to a map g satisfying $g((Z \times (\bigvee_{i=1}^{2M} X_i)) \vee T)^{2n} \subset Z \times (\bigvee_{i=1}^{2M} X_i)$.*

Proof: Since the $(2n-1)$-skeleta of the spaces R_Z and $(Z \times (\bigvee_{i=1}^{2M} X_i))$ coincide, f maps $(R_Z \vee T)^{2n-1}$ into $(Z \times (\bigvee_{i=1}^{2M} X_i))$. We want to prove the existence of a map f' making the following diagram

$$\begin{array}{ccc}
((Z \times (\bigvee_{i=1}^{2M} X_i)) \vee T)^{2n-1} & \xrightarrow{\ f_1\ } & Z \times (\bigvee_{i=1}^{2M} X_i) \\
\downarrow & & \| \\
((Z \times (\bigvee_{i=1}^{2M} X_i)) \vee T)^{2n} & \xrightarrow{\ f'\ } & Z \times (\bigvee_{i=1}^{2M} X_i) \\
\downarrow & & \downarrow \\
R_Z \vee T & \xrightarrow{\ f\ } & R_Z
\end{array}$$

commutative (up to homotopy). In the diagram, the vertical arrows are canonical injections, and f_1 is the restriction of f. The obstruction to the existence of f' is a linear map

$$ob(f_1)\colon H_{2n}\Bigg(\Big(Z \times (\bigvee_i X_i)\Big) \vee T\Bigg) \to \mathbb{Q} \cdot (\alpha + \omega).$$

Suppose that the obstruction is nontrivial. Then for every $2n$-cell of $Z \times (\bigvee_i X_i)$ or of T there is an integer $k \neq 0$ such that the restriction of f_1 to

$$\tilde{f}_1\colon (((Z \times (\bigvee_i X_i)) \vee T)^{2n-1} \cup_\gamma e_1^{2n})_0 \longrightarrow ((Z \times (\bigvee_i X_i)) \cup_{\alpha+\omega} e^{2n})_0 = R_Z$$

satisfies $H_{2n}(\tilde{f}_1)([e_1^{2n}]) - k[e^{2n}] \in H_{2n}(Z \times (\bigvee_i X_i))$.

Let

$$\Omega_1 \in H^{2n}(((Z \times (\bigvee_i X_i)) \vee T)^{2n-1} \cup_\gamma e_1^{2n})$$

be a cohomology class satisfying $\Omega_1([e_1^{2n}]) = 1$. Since $\Omega([e^{2n}]) = 1$, we have $H^{2n}(\tilde{f}_1)(\Omega) = k\Omega_1$.

By construction, e_1^{2n} is a $2n$-cell of either T or $Z \times X_i$. Let S be T or $Z \times X_i$ according to which space this cell is attached to. The composition $\rho \circ \tilde{f}_1|_S$ satisfies $H^{2n}(\rho \circ \tilde{f}_1|_S)(\Omega) \neq 0$ in contradiction with Lemma 6. This contradiction shows that the map $ob(f_1)$ is trivial. ∎

LEMMA 8: *Let $f\colon R_Z \to R_Z$ be a continuous map such that*

$$f\Bigg(Z \times \Big(\bigvee_{i=1}^{2M} X_i\Big)\Bigg)^{2n} \subset Z \times \Big(\bigvee_{i=1}^{2M} X_i\Big).$$

Let $q\colon (Z \times (\bigvee_{i=1}^{2M} X_i))^{2n} \to Z$ be the canonical projection and we suppose that $q \circ f\colon Z^{2n} \to Z$ extends to a homotopy self-equivalence f' of Z. Then $f\colon R_Z \to R_Z$ is also a homotopy self-equivalence.

Proof: Let ψ be a homotopy inverse of $f'\colon Z \to Z$, and let $g = (\psi \times id) \circ f$. Then the following diagram

$$\begin{array}{ccccc}
(Z \times \bigvee_i X_i)^{2n} & \xrightarrow{f} & (Z \times \bigvee_i X_i) & \xrightarrow{\psi \times id} & Z \times \bigvee_i X_i \\
\downarrow & & \downarrow & & \downarrow \\
(Z \times \bigvee_i X_i) \cup_{\alpha+\omega} e^{2n} & \xrightarrow{f} & R_Z & \longrightarrow & ((Z \times \bigvee_i X_i) \cup_{\psi(\alpha)+\omega} e^{2n})_0
\end{array}$$

commutes. We have

$$\pi_{2n-1}(g)(\alpha) = \alpha + \alpha', \quad \text{with } \alpha' \in \pi_{2n-1}(\bigvee_i X_i), \quad \text{and}$$

$$\pi_n(g)(k_i) = l_i + m_i, \quad \text{with } l_i \in \pi_n(Z), \quad m_i \in \pi_n(\bigvee_i X_i),$$

where k_i denotes a generator of $\pi_n(X_i)$. The map $q \circ g$: $(Z \times (\bigvee_i X_i))^{2n} \to Z$ extends to $Z \times (\bigvee_i X_i)$ since $\pi_{\geq 2n}(Z) = 0$. This shows that the Whitehead brackets $[l_i, -]$ are zero in $\pi_*(Z)$. In particular,

$$\pi_{2n-1}(g)(\omega) = \sum_{i=1}^{M} \pi_{2n-1}(g)([k_{2i-1}, k_{2i}]) = \sum_{i=1}^{M} [m_{2i-1}, m_{2i}] \in \pi_{2n-1}\left(\bigvee_{i=1}^{2M} X_i\right).$$

Therefore $\pi_*(g)(\alpha + \omega) = \alpha + \omega'$, with $\omega' \in \pi_{2n-1}(\bigvee_i X_i)$. Since g extends to a map g' from R_Z into $((Z \times \bigvee_i X_i) \cup_{\psi(\alpha)+\omega} e^{2n})_0$, we necessarily have

$$\psi(\alpha) = \nu\alpha, \quad \omega' = \nu\omega, \quad \text{and} \quad H_*(f)[e^{2n}] = \nu[e^{2n}],$$

for some $\nu \neq 0$. In particular, $H^{2n}(f)(\Omega) = \nu\Omega$. It follows that $H^n(f){:}\, H^n(\bigvee_i X_i) \to H^n(R_Z) = H^n(Z) \oplus H^n(\bigvee_i X_i)$ is injective.

We proceed to prove that $H^n(f)$ maps $H^n(\bigvee_i X_i)$ into $H^n(\bigvee_i X_i)$. Suppose this is not the case. Denote by i_0 the least integer i such that $H^n(f)(x_i) \notin H^n(\bigvee_i X_i)$. Then we have

$$H^n(f)(x_i) = y + z, \quad y \in H^n(\bigvee_i X_i), \quad z \in H^n(Z), \quad z \neq 0.$$

Let V be the subspace generated by the x_i with $|i - i_0| \geq 2$. Clearly, $V \cdot x_{i_0} = 0$. On the other hand, for degree reasons, V contains a subspace W of dimension greater than $\frac{3}{2}M - 2$, such that $H^n(f)(W) \subset H^n(\bigvee_i X_i)$. In this case, $\dim H^{2n}(f)(W \cdot x_{i_0}) = \dim(H^n(f)(W) \otimes \mathbb{Q}z) > \frac{3}{2}M - 2$, which is impossible.

Therefore, $H^n(f)$ maps bijectively $H^n(\bigvee_i X_i)$ into itself. We conclude that $H^n(f)$ is an isomorphism, and that $H^n(\bigvee_i X_i)$, $H^*(Z)$ and Ω are in the image of $H^*(f)$. Since $H^*(R_Z)$ is generated by those elements, $H^*(f)$: $H^*(R_Z) \to H^*(R_Z)$ is an isomorphism. ∎

PROPOSITION 13: *The decomposition $R_Z \vee T$ is irreducible relative to T.*

Proof: We first verify the irreducibility property $(P2)$. Let

$$f = \bigvee_k f_k\colon \bigvee_{k \in K} (R_Z \vee T) \to R_Z$$

be an H_2-surjective map. By Lemma 7, each map f_k maps the $2n$-skeleton of $Z \vee T$ into $Z \times (\bigvee_{i=1}^{2M} X_i)$. Since $\pi_{\geq 2n}(Z) = 0$, the map $p \circ f_k|_{Z\vee T}$ extends to a map g_k: $Z \vee T \to Z$.

If $n > 2$, $H_2(g_k) = H_2(f_k)$, and thus $\bigvee_k g_k$ is H_2-surjective. If $n = 2$, by the choice of N, f_k maps each $H_2(X_i)$ into $H_2(\bigvee_i X_i)$. Therefore $\bigvee_k g_k$: $\bigvee_{k\in K}(Z \vee T) \to Z$ is also H_2-surjective.

Now, in both cases, since $Z \vee T$ is irreducible relative to T, there exists k such that $g_k|_Z$: $Z \to Z$ is a homotopy self-equivalence. It follows from Lemma 8 that $f_k|R_Z$: $R_Z \to R_Z$ is also a homotopy self-equivalence.

We now verify the irreducibility property (I_2). Let f: $R_Z \to R_Z$ be a map inducing an isomorphism on H_2. By Lemma 7, f maps the $2n$-skeleton of Z into $Z \times (\bigvee_{i=1}^{2M} X_i)$. Since $\pi_{\geq 2n}(Z) = 0$, the map $p \circ f$ extends to a map g: $Z \to Z$.

If $n > 2$, $H_2(g) = H_2(f)$, and thus g is a homotopy self-equivalence. If $n = 2$, by the choice of N, f maps each $H_2(X_i)$ into $H_2(\bigvee_{=1}^{2M} X_i)$. Therefore, $H_2(f)$: $H_2(Z) \to H_2(Z)$ is a surjective map. Since Z is irreducible, g: $Z \to Z$ is a homotopy self-equivalence.

Finally, in both cases, Lemma 8 implies that f is a homotopy self-equivalence. ∎

6. Proof of Theorem 1

In this section all spaces will be strictly simply-connected c-finite rational spaces. We begin by proving the theorem in a special case.

PROPOSITION 14: *Let $Z\vee Y \overset{s}{\ll} Y$ be a strict inequality with $Z\vee Y$ an irreducible decomposition relative to Y, and Z a hyperbolic space. Then there are infinitely many non-comparable simply-connected finite rational spaces Z_n, $n \geq 1$, such that*

$$Z \vee Y \overset{s}{\ll} Z_n \overset{s}{\ll} Y.$$

Proof: We denote by $(\mathbb{L}(V), d)$ the Quillen model for Z. In this case, the group $Aut(\mathbb{L}(V) \otimes_{\mathbb{Q}} \mathbb{C}, d)$ is a finite dimensional complex algebraic group that acts, for any integer q, in an algebraic way on the complex vector space $H_q(\mathbb{L}(V)\otimes_{\mathbb{Q}}\mathbb{C}, d) = \pi_{q+1}(Z) \otimes \mathbb{C}$.

Let M denote the cohomological dimension of $Z \vee Y$, and N the dimension of the algebraic variety $Aut(\mathbb{L}(V) \otimes_{\mathbb{Q}} \mathbb{C}, d)$. We choose an integer $q > M$ such that dim $\pi_{q+1}(Z) \otimes \mathbb{Q} > N$, and we choose a nonzero element $\alpha_1 \in H_q(\mathbb{L}(V), d)$. In this case $\alpha_1 \cdot Aut(\mathbb{L}(V) \otimes_{\mathbb{Q}} \mathbb{C}, d)$ is a constructible set ([17], p. 33) of dimension less than or equal to N. Therefore, there exists an element $\alpha_2 \in H_q(\mathbb{L}(V), d)$

such that $\alpha_2 \notin \alpha_1 \cdot Aut(\mathbb{L}(V) \otimes_{\mathbb{Q}} \mathbb{C}, d)$. We construct in this way a sequence of elements $(\alpha_1, \alpha_2, \ldots)$ such that for $i \neq j$, $\alpha_i \notin \alpha_j \cdot Aut(\mathbb{L}(V) \otimes_{\mathbb{Q}} \mathbb{C}, d)$. Suppose we have constructed $\alpha_1, \ldots, \alpha_m$ satisfying the above requirement. The union $\bigcup_{i=1}^m \alpha_i \cdot Aut(\mathbb{L}(V) \otimes_{\mathbb{Q}} \mathbb{C}), d)$ is a finite union of constructible sets of dimension less than or equal to N. Since we are in characteristic zero, this union is therefore also a constructible set of dimension less than or equal to N. We then choose $\alpha_{m+1} \in H_q(\mathbb{L}(V), d)$ such that $\alpha_{m+1} \notin \bigcup_{i=1}^m \alpha_i \cdot Aut(\mathbb{L}(V) \otimes_{\mathbb{Q}} \mathbb{C}), d)$.

We denote by a_m the element of $\pi_{q+1}(Z)$ corresponding to α_m by the natural isomorphism $H_q(\mathbb{L}(V), d) \cong \pi_{q+1}(Z)$, and we consider the spaces $Z_m = (Z \cup_{a_m} e^{q+2})$. Since q is greater than the dimension of $Z \vee Y$, by Proposition 8, we have

$$Y \vee Z \overset{s}{\ll} Z_m \overset{s}{\ll} Y.$$

Suppose now that $Z_m \ll Z_n$ for some $n \neq m$. This implies the existence of an H_2-surjective map

$$f = \bigvee_k f_k \colon \bigvee_{k \in K} Z_m \to Z_n.$$

The restriction of f to its skeleton of dimension M is an H_2-surjective map $f\colon \bigvee_k (Z \vee Y) \to Z$. Therefore, for some $k_0 \in K$, $f_{k_0}|_Z\colon Z \to Z$ is a homotopy equivalence. Since f_{k_0} is a map from Z_m to Z_n, we have $\pi_q(f_{k_0})(a_m) = a_n$. This implies that $\alpha_n \in \alpha_m \cdot Aut(\mathbb{L}(V) \otimes_{\mathbb{Q}} \mathbb{C}, d)$, which is impossible by construction of the α_i. ∎

Proof of Theorem 1: Suppose that $X \overset{s}{\ll} T$ is a strict inequality between strictly simply-connected c-finite rational spaces. Clearly $X \sim X \vee T \overset{s}{\ll} T$. We then take an irreducible decomposition of X relative to T,

$$X \sim \bigvee_{i=1}^{n} X_i \vee T.$$

Necessarily in the following sequence of inequalities

$$X \sim \bigvee_{i \geq 1} X_i \vee T \ll \bigvee_{i \geq 2} X_i \vee T \ll \cdots \ll X_n \vee T \ll T,$$

one must be strict. Suppose $X_r \vee (\bigvee_{\geq r+1} X_i \vee T) \overset{s}{\ll} \bigvee_{\geq r+1} X_i \vee T$. Then, we write $Z = X_r$ and $Y = \bigvee_{\geq r+1} X_i \vee T$, and we have

$$Z \vee Y \overset{s}{\ll} Y.$$

When Z is hyperbolic, we apply Proposition 14 to conclude, and when Z is elliptic, the space R_Z comes into play. By Propositions 10 and 11, we have

$$Z \vee Y \overset{s}{\ll} R_Z \vee Y \overset{s}{\ll} Y.$$

By Proposition 12, R_Z is hyperbolic, and by Proposition 13, $R_Z \vee Y$ is an irreducible decomposition relative to Y. We then apply Proposition 14 to the inequality $R_Z \vee Y \overset{s}{\ll} Y$. ∎

7. On the density of the poset $\mathcal{B}$

7.1. Construction of poset injections. Let X be an irreducible hyperbolic simply-connected finite CW-complex that is Bousfield equivalent to the sphere S^2. Let ω be an element as in the statement of Theorem 2. We write $\omega_1 = \omega$, $m = \dim X$, and $c = \mathrm{cat}(X)$. We thus have

(a) $\omega_1 \odot 1 \in \pi_{2q_1}(\Omega X) \odot \mathbb{Q}$;
(b) $2q_1 + 1 > m$; and
(c) $\omega_1 \odot 1 \notin R(X)$.

The Lie ideal generated by $\omega_1 \odot 1$, denoted I^{ω_1}, is therefore infinite dimensional. In particular, $I^{\omega_1}_{even}$ is also infinite dimensional as the following lemma shows.

Lemma 9: *If I is an infinite dimensional Lie ideal in the rational homotopy Lie algebra $\pi_*(\Omega T) \odot \mathbb{Q}$ of a simply-connected finite CW-complex T, then I_{even} is also infinite dimensional.*

Proof: Suppose $\dim I_{even} < \infty$, and denote by r the maximal degree of a homogeneous element of I_{even}. Then $I_{>r} = \bigoplus_{s>r} I_s$ is a graded Lie algebra concentrated in odd degrees. Therefore, $I_{>r}$ is abelian and contained in $R(X)$. Since $R(X)$ is finite dimensional, this would imply I is finite dimensional, which contradicts the hypothesis. ∎

We choose an element $\omega_2 \in \pi_{2q_2}(\Omega X)$ such that $\omega_2 \odot 1 \in I^{\omega_1}$ and

$$\sum_{s<q_2} \dim I^{\omega_1}_{2s} > c+1, \quad \text{and} \quad \omega_2 \odot 1 \notin R(X).$$

In particular, some multiple of ω_2 is in the kernel of the map $\pi_{2q_2}(\Omega X) \to \pi_{2q_2}(\Omega(X \cup_{\tilde{\omega}_1} e^{2q_1+2}))$. We replace ω_2 by this multiple, and we deduce a natural induced map

$$X \cup_{\tilde{\omega}_2} e^{2q_2+2} \to X \cup_{\tilde{\omega}_1} e^{2q_1+2}.$$

The ideal I^{ω_2} generated by $\omega_2 \otimes 1$ is infinite dimensional. Therefore, by Lemma 9, we can choose an element $\omega_3 \in \pi_{2q_3}(\Omega X)$ with $\omega_3 \otimes 1 \in I^{\omega_2}$, $\omega_3 \otimes 1 \notin R(X)$, and

$$\sum_{s<q_3} \dim I_{2s}^{\omega_3} > c+1.$$

Once again we replace ω_3 by some multiple so that ω_3 belongs to the kernel of the map $\pi_{2q_3}(\Omega X) \to \pi_{2q_3}(\Omega(X \cup_{\tilde{\omega}_2} e^{2q_2+2}))$.

We continue by induction and construct a sequence of elements

$$\omega = \omega_1, \omega_2, \ldots, \omega_n, \ldots$$

such that for each $n \geq 2$,

(a) $\omega_n \otimes 1$ belongs to the ideal $I^{\omega_{n-1}}$ generated by $\omega_{n-1} \otimes 1$;
(b) $|\omega_n| = 2q_n$ and $\sum_{s<q_n} \dim I_{2s}^{\omega_{n-1}} > c+1$;
(c) $\omega_n \otimes 1 \notin R(X)$ and
(d) $\omega_n \in \operatorname{Ker}(\pi_{2q_{n-1}}(\Omega X) \to \pi_{2q_{n-1}}(\Omega(X \cup_{\tilde{\omega}_{n-1}} e^{2q_{n-1}+2}))$.

By property (d), the identity map on X extends into a continuous map

$$f_n\colon X_n = (X \cup_{\tilde{\omega}_n} e^{2q_n+2}) \longrightarrow X_{n-1} = (X \cup_{\tilde{\omega}_{n-1}} e^{2q_{n-1}+2}).$$

We obtain in this way an infinite sequence of spaces and maps

$$X \to \cdots \to X_n \xrightarrow{f_n} X_{n-1} \longrightarrow \cdots \xrightarrow{f_2} X_1 = (X \cup_{\tilde{\omega}_1} e^{2q_1+2}).$$

We consider now in more detail the map $f_n\colon X_n \to X_{n-1}$.

LEMMA 10: *The ideal $I^{(1)}$ generated by ω_{n-1} in $\pi_*(\Omega X_n) \otimes \mathbb{Q}$ is infinite dimensional.*

Proof: Let $I = I^{(1)}$. By construction, $I_{2s} = I_{2s}^{\omega_{n-1}}$ for $2s < 2q_n$. The choice of ω_n was made in order to have $\dim I_{even} > c+1$. On the other hand, since X_n is obtained from X by adjonction of a cell, the Lusternik–Schnirelmann category of X_n is less than or equal to $c+1$. Therefore

$$\dim I_{even} > c+1 \geq \operatorname{cat}(X_n).$$

This implies that I is not contained in $R(X_n)$, and is therefore infinite dimensional. ∎

Once again, we choose an element $\alpha_2 \in \pi_{2l_2}(X_n)$ such that

$$\begin{cases} \alpha_2 \otimes 1 \in I_{2l_2}; \\ l_2 > q_n; \\ \alpha_2 \otimes 1 \notin R(X_n); \text{ and} \\ \sum_{r<l_2} \dim I_{2r} > c+2. \end{cases}$$

We then construct the space

$$X_{n-1,2} = (X_n \cup_{\tilde{\alpha}_2} e^{|\alpha_2|+2}).$$

Since $\alpha_2 \otimes 1 \in I$, the class of α_2 in $\pi_{2l_2}(X_{n-1})$ is a torsion element. Replacing α_2 by a multiple, we can assume that $[\alpha_2] = 0$ in $\pi_*(X_{n-1})$.

We denote by $I^{(3)}$ the ideal generated by $\omega_{n-1} \otimes 1$ into $\pi_*(\Omega X_{n-1,2}) \otimes \mathbb{Q}$.

LEMMA 11: *The ideal $I^{(3)}$ is infinite dimensional.*

Proof: By constructiondim $I^{(3)}_{2r} = \dim I_{2r}$, for $r < l_2$. Therefore, $\dim I^{(3)} > c + 2 \geq \mathrm{cat}(X_{n-1,2})$, and thus $I^{(3)}$ is not contained in $R(X_{n-1,2})$. ∎

Let $X_{n-1,3} = (X_{n-1,2} \cup_{\tilde{\alpha}_3} e^{|\alpha_3|+2})$, where α_3 is chosen in $\pi_{2l_3}(\Omega X_{n-1,2})$ with the following properties:

$$\begin{cases} \alpha_3 \otimes 1 \in I^{(3)}; \\ \text{the class of } \alpha_3 \text{ is zero in } \pi_{2l_3}(X_{n-1}); \\ l_3 > l_2; \\ \alpha_3 \otimes 1 \notin R(X_{n-1,2}); \text{ and} \\ \sum_{r<l_3} \dim I^{(3)}_{2r} > c + 3. \end{cases}$$

This process constructs an infinite sequence of spaces and maps

$$X_n = X_{n-1,1} \hookrightarrow X_{n-1,2} \hookrightarrow \cdots X_{n-1,m-1} \xrightarrow{f_{n-1,m}} X_{n-1,m} \cdots$$

defined inductively as follows. The ideal $I^{(m)}$ generated by $\omega_{n-1} \otimes 1$ in the graded Lie algebra $\pi_*(\Omega X_{n-1,m-1}) \otimes \mathbb{Q}$ is infinite dimensional by construction, and we choose an element $\alpha_m \in \pi_{2l_m}(\Omega X_{n-1,m-1})$ such that

$$\begin{cases} \alpha_m \otimes 1 \in I^{(m)}; \\ \text{the class of } \alpha_m \text{ is zero in } \pi_{2l_m}(X_{n-1}); \\ l_m > l_{m-1}; \\ \alpha_m \otimes 1 \notin R(X_{n-1,m-1}); \text{ and} \\ \sum_{r<l_m} \dim I^{(m)}_{2r} > c + m. \end{cases}$$

The last condition forces the ideal $I^{(m+1)}$ to be infinite dimensional. The space $X_{n-1,m}$ is defined by

$$X_{n-1,m} = (X_{n-1,m-1} \cup_{\tilde{\alpha}_m} e^{2l_m+2}).$$

Since the element $[\alpha_m] = 0$ in $\pi_{2l_m}(X_{n-1})$, the identity map on X extends to maps $g_m\colon X_{n-1,m} \to X_{n-1}$, such that $g_m f_{n-1,m} \simeq g_{m-1}$.

An important property of the above construction is that we stay at each step in the category of hyperbolic and irreducible spaces.

PROPOSITION 15: *The spaces X_n and $X_{n,m}$ are hyperbolic and irreducible.*

Proof: The spaces $X_{p,q}$ are hyperbolic by construction because $\pi_*(\Omega X_{p,q}) \otimes \mathbb{Q}$ is infinite dimensional. The spaces $X_{p,q}$ are irreducible because their m-skeleton X is irreducible, and the Hurewicz map h_r: $\pi_r(X_{p,q}) \odot \mathbb{Q} \to H_r(X_{p,q}; \mathbb{Q})$ is zero for $r > m$. ∎

An iterative use of Propositions 8 and 9 shows that the spaces $X_{p,q}$ define different cellular classes. In particular, we have the following sequences of strict inequalities:

$$X \overset{s}{\ll} \cdots \overset{s}{\ll} X_n \overset{s}{\ll} X_{n-1} \overset{s}{\ll} \cdots \overset{s}{\ll} X_2 \overset{s}{\ll} X_1, \quad \text{and}$$
$$X_n = X_{n-1,1} \overset{s}{\ll} X_{n-1,2} \overset{s}{\ll} X_{n-1,3} \overset{s}{\ll} \cdots \overset{s}{\ll} X_{n-1,m} \overset{s}{\ll} \cdots \overset{s}{\ll} X_{n-1}.$$

7.2. PROOF OF THEOREM 2. We use the representation of the rational numbers as finite simple continued fractions. Let

$$[x_1, x_2, \ldots, x_n] := \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{\ddots}{+\cfrac{1}{x_{n-1} + \cfrac{1}{x_n}}}}}.$$

Such an expression is called *a finite simple continued fraction* if all the x_i belong to $\mathbb{N} \backslash \{0\}$. Any simple continued fraction of the above form represents a rational number of $]0,1]$. Conversely, any nonzero rational number in $]0,1]$ can be expressed as a finite simple continued fraction of the above form ([19], Theorem 7.2). Moreover ([19], Theorem 7.1), if $[a_1, \ldots, a_j] = [b_1, \ldots, b_n]$ with $a_j > 1$ and $b_n > 1$, then $j = n$ and $a_i = b_i$, $i = 1, \ldots, n$. On the other hand, we have

$$[a_1, \ldots, a_n, 1] = [a_1, \ldots, a_n + 1].$$

We clearly have the following relations:

$$[a_1, \ldots, a_{2n}] < [a_1, \ldots, a_{2n}, r+1] < [a_1, \ldots, a_{2n}, r]$$
$$< [a_1, \ldots, a_{2n}, 1] = [a_1, \ldots, a_{2n} + 1],$$
$$[a_1, \ldots, a_{2n-1} + 1] = [a_1, \ldots, a_{2n-1}, 1] < [a_1, \ldots, a_{2n-1}, r]$$
$$< [a_1, \ldots, a_{2n-1}, r+1] < [a_1, \ldots, a_{2n-1}].$$

Let us come back to the construction. Starting from (X, ω) we have constructed new pairs $(X_{p,q}, \alpha_{q+1})$. We can start with these pairs and apply the same construction. This defines new spaces with new homotopy classes, and the process can be extended over and over by induction. Let

$$(X,\omega)_{p,q} = (X_{p,q}, \alpha_{q+1}),$$

and define the map $f_{X,\omega}$ by the following inductive process:

$$\begin{aligned}
&f_{X,\omega}(0) = X;\\
&f_{X,\omega}(1) = X \cup_{\tilde{\omega}} e^{2q_1+2} = X_1;\\
&f_{X,\omega}[p] = X_p;\\
&f_{X,\omega}[p,q] = X_{p,q}; \text{ and}\\
&f_{X,\omega}[a_1, a_2, \ldots, a_{2n}, r, s] = f_{(\cdots(((X,\omega)_{a_1,a_2})_{a_3,a_4})\cdots)_{a_{2n-1},a_{2n}}}[r,s].
\end{aligned}$$

This map is well defined because, by construction,

$$X_{p,1} = X_{p+1}$$

and

$$f_{(X,\omega)_{p,q}}[1] = X_{p,q+1}.$$

It follows directly from Propositions 8 and 9 that the map $f_{X,\omega}$ is an injective morphism of posets. ∎

7.3. Example of hyperbolic irreducible finite CW-complexes. For $n \geq 3$, the connected sum of n copies of $P^2(\mathbb{C})$, i.e., $X_n = \#^n P^2(\mathbb{C})$, is an hyperbolic irreducible finite CW-complex. Since $\pi_2(X_n) = \mathbb{Z}^n$, X_n is Bousfield equivalent to S^2. The space X_n is obtained from the wedge of n spheres of dimension two, $S_1^2 \vee S_2^2 \vee \cdots \vee S_n^2$, by adding a four dimensional cell along the element

$$\alpha_n = [a_1, a_1] + [a_2, a_2] + \cdots + [a_n, a_n],$$

where a_r represents the identity map on the sphere S_r^2, while $[-,-]$ denotes the usual Whitehead bracket. Since the cohomology of X_n is not generated as an algebra by only one generator, the element α_n is inert ([12], [10]). This means that the cellular injection

$$S_1^2 \vee S_2^2 \vee \cdots \vee S_n^2 \to X_n$$

induces a surjective map on the rational homotopy groups. Let $q_n\colon X_n \to X_{n-1}$ denote the map obtained by collapsing the sphere S_n^2. The commutativity of the diagram

$$\begin{array}{ccc} S_1^2 \vee S_2^2 \vee \cdots \vee S_n^2 & & \\ \downarrow & \searrow & \\ X_n & \xrightarrow{q_n} & X_{n-1} \end{array}$$

shows that q_n induces a surjective map on the rational homotopy groups when $n \geq 4$. This implies ([10], Theorem 37.3) that $K_n = \operatorname{Ker} \pi_*(\Omega q_n) \otimes \mathbb{Q}$ is a free Lie algebra on at least two generators. We choose an element $\omega_n \in \pi_{\geq 6}(\Omega X_n)$ such that

1. $\omega_n \otimes 1$ is a nonzero element in K_n, and
2. $[\omega_n] = 0$ in $\pi_*(X_{n-1})$.

Clearly $\omega_n \otimes 1 \notin R(X_n)$, and the pair (X_n, ω_n) satisfies the hypothesis of Theorem 2.

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